# MATH 5061 Lecture 4 (Feb 3)

**Last week:** vector fields acting on $C^\infty(M)$. $[\cdot,\cdot]$, $\{\varphi_t\}_t \subseteq \mathrm{Diff}(M)$

$\mathcal{L}_X \alpha := \frac{d}{dt}\big|_{t=0} (\varphi_t)^* \alpha$ for tensors $\alpha$. $(p,q)$-tensors ......

## § Differential Forms

- (Exterior Linear Algebra)

**Goal:** $V$ n-dim'l vector space $\rightsquigarrow \Lambda^k V^*$

**Method 1:** $(k=2)$ $\quad v_1^* \wedge v_2^* = - v_2^* \wedge v_1^*$ $\quad$ (E.g. $v \wedge v = 0$)

**Method 2:** Consider the "skew-symmetrization" operator

$$\mathcal{A} : \otimes^k V^* \longrightarrow \otimes^k V^*.$$

$$\mathcal{A}(\alpha)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

E.g.) $\mathcal{A}(v_1^* \otimes v_2^*) = v_1^* \otimes v_2^* - v_2^* \otimes v_1^*$

**Def<sup></sup>n:** $\alpha \in \otimes^k V^*$ is skew-symmetric if $\mathcal{A}(\alpha) = \alpha$

Denote: $\Lambda^k V^* := \{\alpha \in \otimes^k V^* \text{ skew-symmetric}\} \subseteq \otimes^k V^*$.

"Wedge/exterior Product"

$$\wedge : \Lambda^k V^* \times \Lambda^\ell V^* \longrightarrow \Lambda^{k+\ell} V^*$$

$$\alpha \wedge \beta := \frac{1}{k!\,\ell!} \mathcal{A}(\alpha \otimes \beta)$$

**Exterior Algebra:** $\Lambda^* V^*$

E.g.) $\alpha, \beta \in \Lambda^1 V^* \cong V^*$

$$\Rightarrow \quad (\alpha \wedge \beta)(u,v) = \det \begin{pmatrix} \alpha(u) & \beta(u) \\ \alpha(v) & \beta(v) \end{pmatrix}$$

Properties : (1) $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \quad \forall \alpha \in \Lambda^k V^*, \ \beta \in \Lambda^l V^*$

(2) $\dim \Lambda^k V^* = \binom{\dim V}{k}$ E.g.) $\Lambda^0 V^* \cong \Lambda^n V^* \cong \mathbb{R}$

$\Lambda^1 V^* \cong V^*$ ; $\Lambda^k V^* = 0$ when $k > n$

Why? $\{e_1, \dots, e_n\}$ basis for $V$, $\{e_1^*, \dots, e_n^*\}$ dual basis for $V^*$ $\Rightarrow \{e_{i_1}^* \wedge \dots \wedge e_{i_k}^*\}_{1 \le i_1 < \dots < i_k \le n}$ basis for $\Lambda^k V^*$.

– Apply them to $V = T_pM$, we obtain

Exterior Bundle: $\Lambda^k T^*M := \coprod_{p \in M} \Lambda^k T_p^* M$

$$\Gamma(\Lambda^k T^*M) := \{ k\text{-forms on } M \} =: \Omega^k(M)$$

Properties : (i) $\forall \phi \in \mathrm{Diff}(M)$, $\phi^*(\alpha \wedge \beta) = (\phi^*\alpha) \wedge (\phi^*\beta)$.

(ii) $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta)$.

Exterior Derivative :

$\exists$ linear operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ s.t.

(i) $k=0$ : $(\Omega^0(M) = C^\infty(M))$

$d : C^\infty(M) \to \Omega^1(M)$ agrees with $df$, $\forall f \in C^\infty(M)$

ie $(df)(X) = X(f) \quad \forall f \in C^\infty(M)$

(ii) $d(df) = 0 \quad \forall f \in C^\infty(M)$ ie $d^2 = 0$ on $\Omega^0(M)$.

(iii) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$, $\forall \alpha \in \Omega^k, \beta \in \Omega^l$

<u>FACT:</u> Such an operator exists & is uniquely defined by (i) – (iii).

<u>Reason</u>: locally, $\alpha = \sum_{I=(i_1,\dots,i_k)} \alpha_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(M)$

$$d\alpha = \sum_I d\left(\alpha_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k}\right)$$

$$= \sum_I d\alpha_I \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} + \underbrace{\left(\alpha_I \, \cancelto{0}{d^2 x^{i_1}} \wedge \dots \wedge dx^{i_k} + 0\right)}_{=0}$$

$$= \sum_I \sum_\ell \frac{\partial \alpha_I}{\partial x^\ell} dx^\ell \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

<u>E.g.)</u> $d(x^2 dy \wedge dz) = d(x^2) \wedge dy \wedge dz + 0$

$$= \left[\left(\frac{\partial}{\partial x} x^2\right) dx + \left(\frac{\partial}{\partial y} x^2\right) dy + \left(\frac{\partial}{\partial z} x^2\right) dz\right] \wedge dy \wedge dz$$

$$= 2x \, dx \wedge dy \wedge dz .$$

<u>Properties of d</u>:

(a) $d^2 = 0$ on $\Omega^k(M)$ <u>for all k</u>

<u>Digression</u>: de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

Now, $d^2 = 0 \Rightarrow \ker(d) \supseteq \mathrm{Im}(d)$

defines de Rham cohomology: $H^k_{dR}(M) := \frac{\ker(d)}{\mathrm{Im}(d)}$

(b) $d \circ \phi^* = \phi^* \circ d \quad \forall \phi \in \mathrm{Diff}(M)$.

and $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d \quad \forall X \in \Gamma(TM)$.

(c) <u>Cartan's "Magic" Formula</u>

$$\boxed{\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d} \quad \text{on } \Omega^k(M).$$

"**Proof**": (a) - (b) easy to check.

(c) Check $P_X := d \circ \iota_X + \iota_X \circ d$ satisfies the properties of $\mathcal{L}_X$

**Eg.)** $P_X(f) = d \circ \underbrace{\cancel{\iota_X(f)}}_{0} + \iota_X \circ \overbrace{d(f)}^{df} = df(X) = X(f) = \mathcal{L}_X f$

$\square$

**Cor**: $\forall\, \alpha \in \Omega^1(M)$,

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

**Pf**: Apply Cartan's formula to 1-forms:

$$(\mathcal{L}_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\mathcal{L}_X Y) = X(\alpha(Y)) - \alpha([X, Y])$$

$\parallel$ Cartan

$$(d \circ \iota_X \alpha + \iota_X \circ d\alpha)(Y) = \underbrace{d(\alpha(X))(Y)}_{Y(\alpha(X))} + d\alpha(X, Y)$$

$\square$

## § Volume Forms & Integration

**Def$^{\underline{n}}$**: A volume form on $M^n$ is a nowhere vanishing $n$-form $\omega$

i.e. $\omega \in \Omega^n(M)$, $\omega_p \neq 0$ at each $p \in M$.

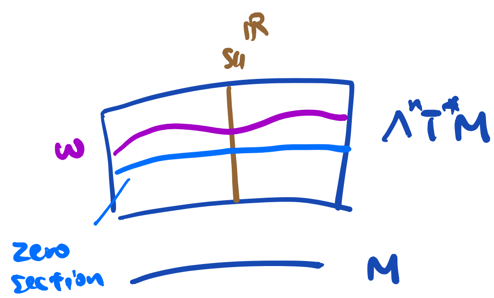
**Thm**: TFAE:

(i) $\exists$ volume form $\omega$ on $M^n$

(ii) $\Lambda^n T^* M$ is a trivial (rank 1) bundle over $M$

(iii) $M$ is orientable.

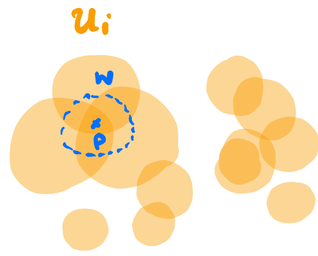
"**Sketch of Proof**": (i) $\Leftrightarrow$ (ii) trivial



(i) $\Rightarrow$ (iii)

locally. $\omega = \overset{\overset{0}{\wedge}}{a}\, dx^1\wedge\cdots\wedge dx^n$

$\left( \begin{array}{l} F: \mathbb{R}^n \to \mathbb{R}^n \text{ diffeo.} \\ F^*(dx^1\wedge\cdots\wedge dx^n) = \det(dF)\, dx^1\wedge\cdots\wedge dx^n \end{array} \right)$

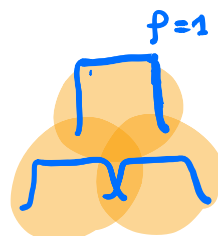



(iii) $\Rightarrow$ (i) Use "Partition of Unity"

$\{\mathcal{U}_i\}_{i\in I}$ "locally finite" open cover of M

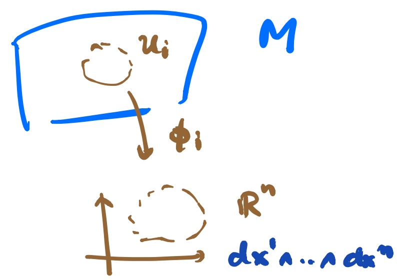
ie $\forall p\in M$, $\exists$ nbd. $W\subseteq M$ of $p$ st $W\cap \mathcal{U}_i \neq \phi$ for finitely many $i\in I$

A partition of unity subordinate to $\{\mathcal{U}_i\}_{i\in I}$ is a family of smooth functions $\rho_i : M \to \mathbb{R}_{\geq 0}$, $i\in I$, s.t.




- $\operatorname{supp} \rho_i \subseteq \mathcal{U}_i \quad \forall i\in I$
- $\underbrace{\sum_{i\in I} \rho_i(p)}_{\text{finite sum}} = 1 \quad \forall p\in M.$

Locally, $dx^1\wedge\cdots\wedge dx^n$ is a volume form on $\mathbb{R}^n$ (loc. defined on M)




$\{(\mathcal{U}_i,\phi_i)\}_{i\in I}$ loc. finite oriented atlas

$\rightsquigarrow$ $\{\rho_i\}_{i\in I}$ partition of unity

$$\omega := \sum_{i\in I} \rho_i\, \phi_i^*(dx^1\wedge\cdots\wedge dx^n)$$

is a volume form.

**Integration** : $\int_M : \Omega^n(M) \to \mathbb{R}$ ; $\int_M \alpha := \sum_{i\in I} \int_{\mathcal{U}_i} \rho_i \alpha = \sum_{i\in I} \overbrace{\int_{\mathcal{U}_i}}^{\text{integration on } \mathbb{R}^n} (\rho_i\circ\phi_i^{-1})(\phi_i^{-1})^*\alpha$

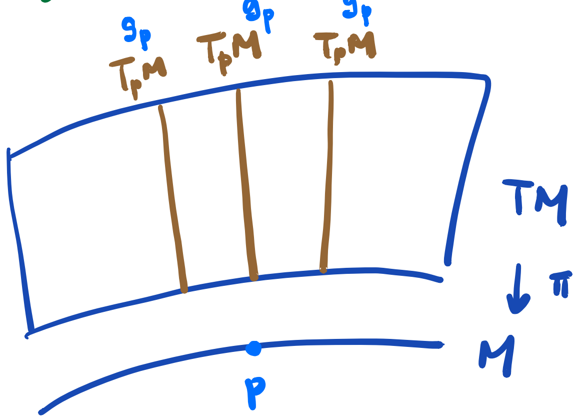
on oriented M

Fixing volume form $\omega$ $\Rightarrow$ $\int_M : C^\infty(M) \to \mathbb{R}$ : $\int_M f := \int_M f\omega$

## § Riemannian metrics

**Def$^{\underline{n}}$**: A Riemannian metric on $M^n$, denoted by $g$ or $\langle , \rangle$,
is an association of an (pos. definite) inner product $g_p$ or $\langle , \rangle_p$ defined
on $T_pM$ depending "smoothly" on $p \in M$.

Picture:




Locally, in coord. $x^1, \dots, x^n$ on $M$.

$$g = (g_{ij}) \quad \text{symm. pos-definite } n \times n \text{ matrices (of fcn)}$$

where $g_{ij}(x^1, \dots, x^n) := g_{(x^1,\dots,x^n)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ depends smoothly on $(x^1, \dots, x^n)$

Equivalently, $g$ is a symmetric $(0,2)$-tensor on $M$
which is pos. definite at every point on $M$.

**Def$^{\underline{n}}$**: $(M^n, g)$ Riemannian manifold

**Def$^{\underline{n}}$**: A smooth map $f : (M, g) \to (N, h)$ between Riem. manifolds
is (i) an isometry if $f^*h = g$ (i.e. $g_p(u,v) = h_{f(p)}(df_p(u), df_p(v))$)

(ii) an local isometry at $p$ if $\exists$ nbd $\mathcal{U}$ of $p$ s.t.

$f : \mathcal{U} \to f(\mathcal{U})$ is an isometry

**Examples:** 1) $\mathbb{R}^n$, $g_{Eucl.} := \sum_{i=1}^n dx^i \otimes dx^i$

2) **Isometric Immersions**: $f : M \to (N, h)$ immersion.

$\Rightarrow$ $f^* h$ is a Riem. metric on $M$

( Why? $(f^*h)_p(u,v) := h_{f(p)}(\underset{1\text{-}1}{df_p}(u), \underset{1\text{-}1}{df_p}(v))$ )

In particular, $M \subset N$, then $\iota : M \hookrightarrow N$ inclusion map.

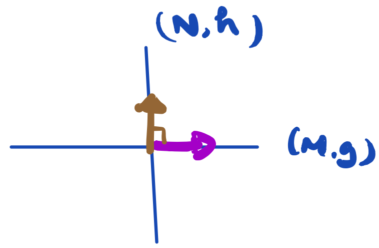
$h$ Riem metric on $N$ $\Rightarrow$ $h|_M := \iota^* h$ induced metric on $M$.

E.g.) $S^n \subseteq (\mathbb{R}^{n+1}, g_{Eucl.})$ $\Rightarrow$ $g_{Eucl.}|_{S^n} =: g_{round}$.

3) **Product metric**: $(M, g)$, $(N, h)$ Riem mfd

$\rightsquigarrow$ $(M \times N, g \oplus h)$ product Riem. mfd.

locally, $(g \oplus h)_{ij} = \left(\begin{array}{c|c} g_{ij} & 0 \\ \hline 0 & h_{ij} \end{array}\right)$



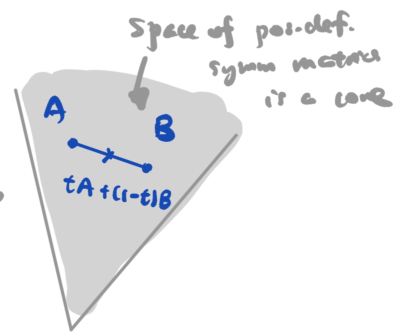

E.g.) Consider $S^1 \subseteq (\mathbb{R}^2, g_{Eucl.})$

$\rightsquigarrow$ $T^n := \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$ flat $n$-torus

**Prop**: Every smooth manifold $M^n$ admits a Riem. metric.

"**Sketch of Proof**": $\{(U_i, \phi_i)\}$ atlas, w. P.O.U. $\{\rho_i\}$.

define: $g := \sum_{i \in I} \rho_i \underbrace{\left( \sum_{j=1}^n dx^j \otimes dx^j \right)}_{\text{pos definite}}$




$\square$

**Remark:** Every oriented $(M^n, g)$ has a preferred volume form

$$\omega = dV_g \overset{\text{loc.}}{=} \underbrace{\sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n}_{\text{invar. under change of coord.}}$$

**Remark:** Let $c : [a,b] \to M$ smooth curve.

Define $\text{Length}(c) := \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}\, dt$

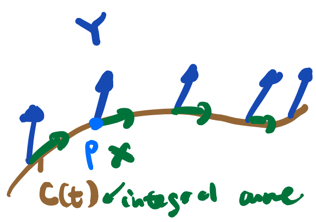
## § Connections

**Q:** How do we differentiate $f \in C^\infty(M)$?

**A:** $f \in C^\infty(M) \rightsquigarrow \underbrace{X(f) \text{ or } df(X) \text{ or } \mathcal{L}_X f}_{\text{the same!}}$

**Q:** How do we differentiate vector fields $Y \in \Gamma(TM)$?

**A1:** Lie derivative $\mathcal{L}_X Y = [X, Y]$




differentiate $Y$ along $c$?

$(\mathcal{L}_X Y)(p)$ depends on the values of $X$ and $Y$ in a nbd. of $p$

**Reason:** $\mathcal{L}_X Y$ is NOT tensorial in $X$.

$$\mathcal{L}_{fX} Y \neq f\, \mathcal{L}_X Y$$

**A2:** "Covariant derivative" $\nabla_X Y$

needs an extra structure of a "connection".

**Def$^{\underline{n}}$:** An affine connection on $M$ is a map

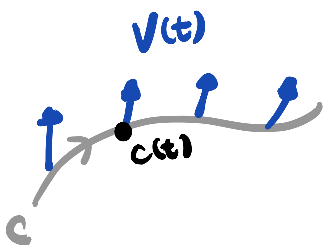
$$\nabla : \mathcal{T}(TM) \times \mathcal{T}(TM) \longrightarrow \mathcal{T}(TM)$$
$$(X, Y) \longmapsto \nabla_X Y$$

s.t. (i) $\nabla$ is bilinear over $\mathbb{R}$.

(ii) $\nabla_{fX} Y = f \nabla_X Y \quad \forall f \in C^\infty(M)$

(iii) $\nabla_X (fY) = X(f)\, Y + f\, \nabla_X Y \quad \forall f \in C^\infty(M)$

One can use $\nabla$ to define a covariant derivative as follows:




$c(t) : I \to M$ curve

$V(t) : I \to TM$ vector field along $c$.

i.e. $V(t) \in T_{c(t)}M$

**Def$^{\underline{n}}$:** $\dfrac{DV}{dt} := \nabla_{\underbrace{c'}_{\text{a vector field along } c}} V$

Covariant derivative of $V$ along $c$

$\because$ $\nabla$ is tensorial in $X$-variable

$\Rightarrow$ well-definedness.